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A NEW NON-LINEAR RECURRENCE IDENTITY CLASS FOR HORADAM SEQUENCE TERMS

Peter J. Larcombe and Eric J. Fennessey

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Abstract We state, and prove by a succinct matrix method, a non-linear recurrence identity class for terms of the so called Horadam sequence. A particular instance was established (in equivalent form) over half a century ago by A.F. Horadam, which provides a starting point for the discussion and an introduction to our formulation technique.

1 Introduction

Denote by $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$, in standard format, the four-parameter Horadam sequence arising from the second order linear recursion

$$w_{n+2} = pw_{n+1} - qw_n, \qquad n \geq 0, \tag{1.1}$$

for which $w_0 = a$ and $w_1 = b$ are initial values; many familiar sequences are derived as special cases of this most general recurrence. From two seminal papers produced by A.F. Horadam in the 1960s (where the notation $w_n(a, b; p, q)$ and the form of (1.1) were fixed), work has been conducted on the so called Horadam sequence for many decades since. Recent survey articles attempt to chart such activity accordingly (see [5, 4] from 2013 and 2017, respectively), in between which a posthumous tribute to Horadam—who passed away in 2016—has also been published [3].

2 A Result and Proof

We begin by stating, and proving by a succinct matrix method, an identity first seen (in equivalent form) in Horadam's earliest work on his eponymous sequence.

2.1 The Result

Consider the following result, which has been checked by computer (see Section 2.3 for more details).

Identity (Horadam). For $r, t \geq 0$,

$$w_{r+1}w_{t+1} - qw_r w_t = bw_{r+t+1} - qaw_{r+t}.$$

Example. Consider, for $r = 1, t = 2$ (using (1.1) as needed), the identity r.h.s. $= bw_4 - qaw_3 = b(pw_3 - qw_2) - qaw_3 = (bp - qa)w_3 - bqw_2 = (w_1 p - qw_0)w_3 - w_1 qw_2 = w_2 w_3 - qw_1 w_2 =$ l.h.s.

2.2 The Proof

Proof. Let

$$\mathbf{H}(p, q) = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \tag{I.1}$$

from which the recursion (1.1) readily delivers the matrix power relation

$$\begin{pmatrix} w_n \\ w_{n-1} \end{pmatrix} = \mathbf{H}^{n-1}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \tag{I.2}$$

that holds for $n \geq 1$. Writing

$$\mathbf{D}(q) = \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}, \tag{I.3}$$

then

$$\mathbf{T}_n(q) = \mathbf{D}(q)\mathbf{H}^{n-1}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} = \begin{pmatrix} w_n \\ -qw_{n-1} \end{pmatrix} \tag{I.4}$$

from (I.2),(I.3), so that

$$(w_1, w_0)\mathbf{T}_n(q) = w_1 w_n - q w_0 w_{n-1} \tag{I.5}$$

and, in particular,

$$(w_1, w_0)\mathbf{T}_{r+t+1}(q) = w_1 w_{r+t+1} - q w_0 w_{r+t} = b w_{r+t+1} - q a w_{r+t}. \tag{I.6}$$

Our proof relies on the neat (quasi-commutativity) result

$$\mathbf{D}(q)\mathbf{H}(p,q) = [\mathbf{H}(p,q)]^T \mathbf{D}(q) \tag{I.7}$$

satisfied by the matrices $\mathbf{D}(q)$ and $\mathbf{H}(p,q)$ (T is the transpose operator), to which we make appeal, as required, along with (I.2). Beginning with the definition of $\mathbf{T}_n(q)$ given in (I.4), we now write

$$\begin{aligned}
(w_1, w_0)\mathbf{T}_{r+t+1}(q) &= (w_1, w_0)\mathbf{D}(q)\mathbf{H}^{r+t}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= (w_1, w_0)\mathbf{D}(q)\mathbf{H}^{t+r}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= (w_1, w_0)[\mathbf{H}^{t+r}(p,q)]^T \mathbf{D}(q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= (w_1, w_0)[\mathbf{H}^{t}(p,q)\mathbf{H}^{r}(p,q)]^T \mathbf{D}(q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= (w_1, w_0)[\mathbf{H}^{r}(p,q)]^T[\mathbf{H}^{t}(p,q)]^T \mathbf{D}(q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= \begin{pmatrix} w_1 \\ w_0 \end{pmatrix}^T [\mathbf{H}^{r}(p,q)]^T \mathbf{D}(q)\mathbf{H}^{t}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= \left[\mathbf{H}^{r}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix}\right]^T \mathbf{D}(q)\mathbf{H}^{t}(p,q) \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \\
&= \begin{pmatrix} w_{r+1} \\ w_r \end{pmatrix}^T \mathbf{D}(q) \begin{pmatrix} w_{t+1} \\ w_t \end{pmatrix} \\
&= (w_{r+1}, w_r) \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix} \begin{pmatrix} w_{t+1} \\ w_t \end{pmatrix} \\
&= w_{r+1} w_{t+1} - q w_r w_t,
\end{aligned} \tag{I.8}$$

whence the proof is immediate upon equating the r.h.s. expressions of (I.6) and (I.8). □

2.3 Verification and Context

We have validated the identity, as mentioned, by using a selection of r, t values while keeping a, b, p, q symbolic. Alternatively, based on the characteristic polynomial $\lambda^2 - p\lambda + q$ associated with (1.1), those well known closed (Binet) forms of the Horadam sequence general term can be used to algebraically certify it in (i) the non-degenerate characteristic roots case, for which (with distinct roots $\alpha(p,q) = (p + \sqrt{p^2 - 4q})/2$, $\beta(p,q) = (p - \sqrt{p^2 - 4q})/2$ $(p^2 \neq 4q)$)

$$w_n(a,b;p,q) = w_n(\alpha(p,q), \beta(p,q), a, b) = \frac{(b - a\beta)\alpha^n - (b - a\alpha)\beta^n}{\alpha - \beta}, \qquad n \geq 0, \tag{2.1}$$

and (ii) the degenerate characteristic roots case, for which (with non-distinct roots $\alpha(p) = \beta(p) = p/2$ $(p^2 = 4q)$)

$$w_n(a,b;p,p^2/4) = w_n(\alpha(p), a, b) = bn\alpha^{n-1} - a(n-1)\alpha^n, \qquad n \geq 0; \tag{2.2}$$

we have done this also.

We finish by noting that the identity first occurred (in an equivalent form, see the Appendix) in both of Horadam's aforementioned papers which were published in 1965; it is merely listed in his *Fibonacci Quarterly* paper [1, (4.1), p. 171], and derived as [2, (25), p. 440] in a *Duke Mathematical Journal* article (where we see that the formulation is quite different to ours here, utilising the ordinary generating function of the Horadam sequence).

It is clear that our methodology can be applied to linear recursions of degree three or more to derive identities for the terms therefrom; work continues on this line of enquiry.

3 A Generalised Identity

The matrix $\mathbf{D}(q)$ (I.3) is but one of an infinite number of matrices that exhibit quasi-commutativity with $\mathbf{H}(p,q)$ (I.1). Writing

$$\mathbf{S}(s_1, s_2, s_3, s_4) = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, \tag{3.1}$$

then the condition $\mathbf{S}(s_1, s_2, s_3, s_4)\mathbf{H}(p,q) = [\mathbf{H}(p,q)]^T \mathbf{S}(s_1, s_2, s_3, s_4)$ results in four equations in the elements of $\mathbf{S}$ whose solution (a straightforward reader exercise) allows a general form of $\mathbf{S}$ to be determined. It is found that, for arbitrary β, γ,

$$\mathbf{S}^{[\beta,\gamma]}(p,q) = \begin{pmatrix} \gamma & \beta \\ \beta & -(\beta p + \gamma q) \end{pmatrix}. \tag{3.2}$$

Omitting the details, then if the same proof procedure is followed as before (with $\mathbf{S}^{[\beta,\gamma]}(p,q)$ replacing $\mathbf{D}(q)$) a previously unseen result is yielded (this has also been algebraically verified) of which Horadam's identity is the $\beta = 0$, $\gamma = 1$ instance (with $\mathbf{S}^{[0,1]}(p,q) = \mathbf{D}(q)$); since β, γ are arbitrary, we have a *complete class of infinite cases* available to us.

Identity (Generalised). For $r \geq 0$, $t \geq 1$, and arbitrary constants β, γ,

$$w_{r+1}(\gamma w_{t+1} + \beta w_t) - q w_r(\gamma w_t + \beta w_{t-1}) = \gamma b w_{r+t+1} + (\beta b - \gamma q a) w_{r+t} - \beta q a w_{r+t-1}.$$

Note, of course, that for $\beta \neq 0$ the result can be characterised by a single arbitrary parameter $\zeta = \gamma/\beta$ (in addition to the recurrence variables q, a and b delivering the sequence $\{w_n(a,b;p,q)\}_0^\infty$).

Appendix

For completeness, we note that Horadam's version of our (non-generalised) identity, namely,

$$w_m w_n - q w_{m-1} w_{n-1} = a w_{m+n} + (b - pa) w_{m+n-1}, \tag{A.1}$$

is an equivalent one by using (1.1) to write

$$\begin{aligned} w_m w_n - q w_{m-1} w_{n-1} &= a(w_{m+n} - p w_{m+n-1}) + b w_{m+n-1} \\ &= a(-q w_{m+n-2}) + b w_{m+n-1} \\ &= b w_{m+n-1} - q a w_{m+n-2}, \end{aligned} \tag{A.2}$$

and setting $m = r + 1$, $n = t + 1$.

References

[1] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, *Fib. Quart.* **3**, 161–176 (1965).

[2] A. F. Horadam, Generating functions for powers of a certain generalised sequence of numbers, *Duke Math. J.* **32**, 437–446 (1965).

[3] P. J. Larcombe, Alwyn Francis Horadam, 1923–2016: a personal tribute to the man and his sequence, *Bull. I.C.A.* **78**, 93–107 (2016).

[4] P. J. Larcombe, Horadam sequences: a survey update and extension, *Bull. I.C.A.* **80**, 99–118 (2017).

[5] P. J. Larcombe, O. D. Bagdasar and E. J. Fennessey, Horadam sequences: a survey, *Bull. I.C.A.* **67**, 49–72 (2013).

Author information

Peter J. Larcombe, Department of Computing and Mathematics, College of Engineering and Technology, University of Derby, Kedleston Road, Derby DE22 1GB, U.K.
E-mail: p.j.larcombe@derby.ac.uk

Eric J. Fennessey, BAE Systems Surface Ships Ltd., HM Naval Base, Portsmouth PO1 3NJ, U.K.
E-mail: eric.fennessey@baesystems.com

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